Published by Institute of Physics Publishing for SISSA/ISAS
Received: *May 29, 2003*
Revised: *July 4, 2003*
Accepted: *July 14, 2003*

The scaling equation of state of the 3-D O(4) universality class

Francesco Parisen Toldin
Scuola Normale Superiore
Piazza dei Cavalieri 7, I-56126 Pisa, Italy
E-mail: f.parisentoldin@sns.it

Andrea Pelissetto
Dipartimento di Fisica dell'Università di Roma "La Sapienza" and INFN
Piazzale Moro 2, I-00185 Roma, Italy
E-mail: Andrea.Pelissetto@roma1.infn.it

Ettore Vicari
Dipartimento di Fisica dell'Università di Pisa and INFN
Via Buonarroti 2, I-56127 Pisa, Italy
E-mail: vicari@df.unipi.it

Abstract: We determine the scaling equation of state of the three-dimensional O(4) universality class, which is relevant for the finite-temperature transition of quantum chromodynamics with two light flavors. We first consider the small-field expansion of the effective potential (Helmholtz free energy). Then, we apply a systematic approximation scheme based on polynomial parametric representations that are valid in the whole critical regime, satisfy the correct analytic properties (Griffiths' analyticity), take into account the Goldstone singularities at the coexistence curve, and match the small-field expansion of the effective potential. From the approximate representations of the equation of state, we obtain estimates of several universal amplitude ratios.

Keywords: Nonperturbative Effects, QCD, Lattice QCD.

Contents

1. Introduction

In the theory of critical phenomena continuous phase transitions can be classified into universality classes determined only by a few properties characterizing the system, such as the space dimensionality, the range of interaction, the number of components of the order parameter, the symmetry and the symmetry-breaking pattern. Renormalization-group theory predicts that critical exponents and scaling functions are the same for all systems belonging to a given universality class. Here we consider the three-dimensional O(4) universality class, which is characterized by a four-component order parameter, O(4) symmetry, and effective short-range interactions. A representative of this universality class is the lattice spin model

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j - \sum_i \vec{H} \cdot \vec{s}_i \, , \tag{1.1}$$

where $\vec{s}_i$ are four-component unit spins and the summation is extended over all nearest-neighbor pairs $\langle ij \rangle$. We are interested in the scaling equation of state that relates the magnetization $\vec{M} \equiv \frac{1}{V}\langle \sum_i \vec{s}_i \rangle$, the temperature T, and a uniform external (magnetic) field $\vec{H}$ near the critical point $T = T_c$ and $\vec{H} = 0$.

The three-dimensional O(4) model is relevant for the finite-temperature behavior of quantum chromodynamics (QCD) with two light-quark flavors. Indeed, according to universality arguments, if two-flavor QCD undergoes a continuous transition, its critical behavior is essentially determined by a few general properties, such as the effective dimensionality at T_c that is three, and the symmetry-breaking pattern that is $O(4) \to O(3)$. Such a breaking pattern is realized in the three-dimensional O(4) universality class. Therefore, the O(4) theory should provide the asymptotic critical behavior at T_c [1, 2, 3], see also the reviews [4, 5]. However, universality arguments do not exclude that the transition is of first order. For example, a first-order transition should be expected if the breaking of the axial $\mathrm{U}(1)_A$ symmetry is effectively weak at T_c [1, 6]. Information on the finite-temperature phase transition in two-flavor QCD has also been achieved by numerical Monte Carlo lattice studies. Recent lattice simulations have provided a rather convincing evidence of a continuous transition at $T_c \approx 172\,\mathrm{MeV}$, see, e.g., refs. [7]–[12], whose scaling properties are substantially consistent with those predicted by the 3-d O(4) universality, although a conclusive evidence in favor of an O(4) scaling behavior in the continuum limit has not been achieved yet.

Since the transition is apparently continuous, the critical behavior around T_c should be described by the 3-d O(4) universality class. In two-flavor QCD the order-parameter field can be written as a four-component vector $\vec{\varphi}$, whose expectation value $\vec{M}$ is related to the quark condensate $\langle \bar{q}q \rangle$, and which is coupled to a uniform external field $\vec{H}$ related to the quark masses. Since the quark masses act as an external (magnetic) field, a quark-mass term smooths out the singularity and instead of the transition one observes an analytic crossover, as in the O(4) model for nonvanishing magnetic field. In the small quark-mass regime the universal features of this analytic crossover can be described by the scaling O(4) theory. The 3-d O(4) scaling equation of state gives the asymptotic relations, for $T \to T_c$ and in the limit of vanishing quark masses, among the reduced temperature $t \equiv T/T_c - 1$, the quark masses, and the quark condensate $\langle \bar{q}q \rangle$. It can be written as

$$\begin{aligned} \vec{M} &= B_c \vec{H} |H|^{(1-\delta)/\delta} E(y)\,, \\ y &= (B/B_c)^{1/\beta} t |H|^{-1/(\beta+\gamma)}\,, \end{aligned} \tag{1.2}$$

where β, δ and γ are the critical exponents, B and B_c are nonuniversal constants fixing the normalizations (they are the amplitudes of the magnetization at the coexistence curve and along the critical isotherm, respectively), and $E(y)$ is a universal function. Similar asymptotic relations can also be written for the (quark-mass) susceptibilities; they can be easily derived by taking appropriate derivatives of the relation (1.2).

The critical equation of state of $O(N)$ models has already been studied in $\epsilon \equiv 4-d$ expansion at two loops [13] and in $1/N$ expansion at order $1/N$ [14]. However, these results do not allow us to obtain a quantitatively accurate determination of the critical equation of state, essentially because the available expansions are too short. On the other hand, on the numerical side, rather accurate results have been obtained from Monte Carlo simulations of the O(4) spin model (1.1) [15, 16, 17].

In this paper we compute the critical equation of state of the 3-d O(4) universality class by using a different analytic method. The starting point is the small-field expansion of the effective potential (in statistical-mechanics language it corresponds to the small-magnetization expansion of the Helmholtz free energy), or equivalently of the equation of state, in the symmetric (high-temperature) phase ($t > 0$). The first few nontrivial terms of this expansion have already been determined by exploiting field-theoretical methods in the continuum φ^4 theory [18, 19]. We use them to determine approximations of the equation of state that are valid in the whole critical regime. This requires an analytic continuation from the high-temperature phase ($t > 0$) to the the coexistence curve ($t < 0$), which can be achieved by using parametric representations [20, 21, 22, 23] implementing in a rather simple way the known analytic properties of the equation of state (Griffiths' analyticity). We construct systematic approximation schemes based on polynomial parametric representations of the critical equation of state that have the correct analytic properties, take into account the Goldstone singularities at the coexistence curve, and match the known small-field behavior of the effective potential. Using the results for the critical equation of state, we then derive estimates of several universal amplitude ratios. As we shall see, we will provide rather accurate determinations of the scaling functions related to the magnetization, the longitudinal (quark-mass) susceptibility, and of several universal amplitude ratios, involving also quantitites defined at the so-called pseudo-critical line that corresponds to the maximum of the longitudinal susceptibility at fixed H.

The paper is organized as follows. In section 2 we discuss the general properties of the critical equation of state. In particular, we consider the small-field expansion of the effective potential and of the critical equation of state in the symmetric phase. In section 3 we describe our approximation schemes based on polynomial parametric representations, and determine the scaling functions that give the scaling behavior of the magnetization (quark condensate) and of the longitudinal susceptibility. Finally, in section 4 we determine several universal amplitude ratios.

2. The scaling equation of state

2.1 General properties

The equation of state relates the magnetization $\vec{M}$, the magnetic field $\vec{H}$, and the reduced temperature $t \equiv (T - T_c)/T_c$. In the neighborhood of the critical point $t = 0$, $H = 0$, it is usually written in the scaling form

$$\vec{H} = B_c^{-\delta} \vec{M} M^{\delta-1} f(x) , \qquad x \equiv t \left(\frac{M}{B}\right)^{-1/\beta} , \tag{2.1}$$

where $M \equiv |\vec{M}|$, $f(x)$ is a universal scaling function such that that $f(-1) = 0$, $f(0) = 1$, and B_c and B are the amplitudes of the magnetization on the critical isotherm and at the coexistence curve:

$$M = B_c H^{1/\delta} , \qquad t = 0 , \tag{2.2}$$

$$M = B(-t)^{\beta} , \qquad t < 0 , \quad H \to 0 . \tag{2.3}$$

The scaling function $f(x)$ and $E(y)$, cf. eq. (1.2), are clearly related:

$$E(y) = f(x)^{-1/\delta} , \qquad y = x f(x)^{-1/(\beta+\gamma)} . \tag{2.4}$$

Here δ, β and γ are critical exponents of the 3-d O(4) universality class. They have been determined rather accurately by exploiting field-theoretical methods and lattice techniques. We mention the field-theory estimates of ref. [24], $\nu = 0.741(6)$ and $\eta = 0.035(5)$, and the Monte Carlo results of ref. [25], $\nu = 0.749(2)$ and $\eta = 0.0365(10)$. A more complete list of

JHEP07(2003)029

references and results can be found in ref. [26]. The other exponents can be obtained by using scaling and hyperscaling relations: $\alpha = 2 - 3\nu$, $\gamma = \nu(2-\eta)$, $\beta = \nu(1+\eta)/2$ and $\delta = (5-\eta)/(1+\eta)$. For example, using the most precise Monte Carlo results one obtains $\alpha = -0.247(6)$, $\gamma = 1.471(4)$, $\beta = 0.388(1)$, and $\delta = 4.789(6)$. Corrections to the scaling behaviors (1.2) and (2.1) are suppressed by powers of t and $|H|$. The leading ones are of order t^{Δ} on the axis $H = 0$ and of order $|H|^{\Delta/(\beta+\gamma)}$ on the critical isotherm (in general corrections have the scaling behavior $t^{\Delta} g(y)$), where $\Delta \approx 0.58$ [24, 25].

The equation of state is analytic for $|H| > 0$, and therefore $f(x)$ is regular everywhere for $x > -1$. In particular, $f(x)$ has a regular expansion in powers of x around $x = 0$

$$f(x) = 1 + \sum_{n=1}^{\infty} f_n^0 x^n \,, \tag{2.5}$$

and a large-x expansion of the form

$$f(x) = x^{\gamma} \sum_{n=0}^{\infty} f_n^{\infty} x^{-2n\beta} \,. \tag{2.6}$$

At the coexistence curve, i.e. for $x \to -1$, the Goldstone singularities appear. General arguments predict that, at the coexistence curve and in three dimensions, transverse and longitudinal susceptibilities behave respectively as

$$\chi_T = \frac{M}{H}\,, \qquad \chi_L = \frac{\partial M}{\partial H} \sim H^{-1/2} \,. \tag{2.7}$$

In particular, the singularity of χ_L for $t < 0$ and $H \to 0$ is governed by the zero-temperature infrared-stable gaussian fixed point [14, 27, 28], leading to the prediction

$$f(x) \approx f_2^{\rm coex} \, (1+x)^2 \qquad \text{for } x \to -1 \,. \tag{2.8}$$

The nature of the corrections to the behavior (2.8) is less clear. It has been conjectured [29, 30, 28], using essentially ϵ-expansion arguments, that, for $x \to -1$, i.e., near the coexistence curve, $v \equiv 1 + x$ has a double expansion in powers of $w \equiv H M^{-\delta}$ and $w^{(d-2)/2}$. This would imply that in three dimensions $f(x)$ could be expanded in integer powers of v at the coexistence curve. On the other hand, an explicit calculation [31] to next-to-leading order in the $1/N$ expansion shows the presence of logarithms in the asymptotic expansion of $f(x)$ for $x \to -1$, so that $f(x)$ cannot be expanded in powers of v. However, these nonanalytic terms represent small corrections of order $O(v^2 \log v)$ compared to the leading behavior (2.8).

Finally, beside the scaling functions $E(y)$ and $f(x)$, we introduce a scaling function associated with the longitudinal susceptibility, by writing

$$\chi_L = B_c |H|^{1/\delta - 1} D(y) \,, \tag{2.9}$$

where

$$D(y) = \frac{1}{\delta}\left[E(y) - \frac{y}{\beta} E'(y)\right] = \frac{\beta f(x)^{1-1/\delta}}{\beta\delta f(x) - x f'(x)} \,. \tag{2.10}$$

The function $D(y)$ has a maximum at $y = y_{\rm max}$ corresponding to the crossover or pseudo-critical line $t_{\rm max}(H)$, see also section 4.

JHEP07(2003)029

2.2 The small-field expansion of the effective potential

Let us consider the continuum φ^4 theory that describes the three-dimensional O(4) universality class, i.e.

$$S = \int d^3x \left\{ \frac{1}{2}\partial_\mu \vec{\varphi}(x) \cdot \partial_\mu \vec{\varphi}(x) + \frac{1}{2} r\, \vec{\varphi}(x) \cdot \vec{\varphi}(x) + \frac{1}{4!} u \left[\vec{\varphi}(x) \cdot \vec{\varphi}(x)\right]^2 - \vec{H} \cdot \vec{\varphi}(x) \right\}, \quad (2.11)$$

where $\vec{\varphi}(x)$ is a four-component real field. Let us also consider a zero-momentum renormalization scheme [32] (see also refs. [20, 33]):

$$\Gamma^{(2)}_{ab}(p) = \delta_{ab} Z_\varphi^{-1} \left[m^2 + p^2 + O(p^4)\right], \quad (2.12)$$

$$\Gamma^{(4)}_{abcd}(0) = Z_\varphi^{-2} m \frac{g}{3} \left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right), \quad (2.13)$$

where $\Gamma^{(n)}_{a_1,\ldots,a_n}$ are n-point one-particle irreducible correlation functions, m and g are the zero-momentum mass scale and quartic coupling, respectively.

The effective potential (Helmholtz free energy) is related to the (Gibbs) free energy of the model. If $\vec{M} \equiv \langle \vec{\varphi} \rangle$ one defines

$$\mathcal{F}(M) = \vec{M} \cdot \vec{H} - \frac{1}{V} \log Z(H), \quad (2.14)$$

where $Z(H)$ is the partition function and the dependence on the temperature is always understood in the notation. The effective potential $\mathcal{F}(M)$ is the generator of the zero-momentum one-particle irreducible correlation functions. In the high-temperature phase it admits an expansion around $M = 0$:

$$\Delta\mathcal{F} \equiv \mathcal{F}(M) - \mathcal{F}(0) = \sum_{j=1}^{\infty} \frac{1}{(2j)!} a_{2j} M^{2j}. \quad (2.15)$$

This expansion can be rewritten in terms of the expectation value of the renormalized field $\varphi_r(x) = Z_\varphi^{-1/2} \varphi(x)$, i.e. $\vec{M}_r \equiv \langle \vec{\varphi}_r \rangle$,

$$\Delta\mathcal{F} = \frac{1}{2} m^2 M_r^2 + \sum_{j=2} m^{3-j} \frac{1}{(2j)!} g_{2j} M_r^{2j}. \quad (2.16)$$

Note that g_4 is the renormalized coupling g appearing in eq. (2.13); its critical limit is the fixed point of the renormalization-group equations, that is the zero of the Callan-Symanzik β-function $\beta(g) \equiv m\partial g/\partial m$. The coefficients g_{2j} approach universal constants (which we indicate with the same symbol) for $m \to 0$. By performing a further rescaling

$$M_r^2 = \frac{m}{g_4} z^2 \quad (2.17)$$

in eq. (2.16), the effective potential can be written as

$$\Delta\mathcal{F} = \frac{m^3}{g_4} A(z), \quad (2.18)$$

where

$$A(z) = \frac{1}{2}z^2 + \frac{1}{4!}z^4 + \sum_{j=3} \frac{1}{(2j)!} r_{2j} z^{2j}\,, \tag{2.19}$$

and

$$r_{2j} = \frac{g_{2j}}{g_4^{j-1}} \qquad j \geq 3\,. \tag{2.20}$$

The universal quantities g_4 and the first few r_{2j} have been estimated by using field-theoretical methods. Fixed-dimension perturbative calculations provide the estimates $g_4 = 17.30(6)$ (from an analysis [24] of the six-loop expansion of $\beta(g)$ [34]), $r_6 = 1.81(3)$ (from an analysis [18] of the corresponding four-loop series [19]) and $r_8 = 0.456$ (three-loop series [19]), while ϵ-expansion computations give [18] $g_4 = 17.5(3)$ (four-loop series), $r_6 = 1.780(8)$ and $r_8 = 0.2(4)$ (three-loop series). Other substantially less precise results and references can be found in refs. [18].

Since $z \propto t^{-\beta}M$ where $t \equiv T/T_c - 1 \propto r - r_c$, the equation of state can be written in the form

$$\vec{H} = \frac{\partial \mathcal{F}(M)}{\partial \vec{M}} \propto \frac{\vec{M}}{|M|} t^{\beta\delta} F(z)\,, \tag{2.21}$$

with

$$F(z) \equiv \frac{\partial A}{\partial z} = z + \frac{1}{6}z^3 + \sum_{j=3} \frac{1}{(2j-1)!} r_{2j} z^{2j-1}\,. \tag{2.22}$$

The two functions $f(x)$ and $F(z)$ are related:

$$z^{-\delta}F(z) = F_0^\infty f(x)\,, \qquad z = z_0 x^{-\beta}\,, \tag{2.23}$$

where $z_0 = (R_4^+)^{1/2}$ is a universal constant, see section 4. Because of Griffiths' analyticity, $\mathcal{F}(M)$ has also a regular expansion in powers of t for $|M|$ fixed. Therefore, $F(z)$ has the large-z expansion

$$F(z) = z^\delta \sum_{k=0} F_k^\infty z^{-k/\beta}\,. \tag{2.24}$$

3. Parametric representations of the equation of state

3.1 Polynomial approximation schemes

In order to obtain a representation of the equation of state that is valid in the whole critical region, we need to extend analytically the expansion (2.19) to the low-temperature region $t < 0$. For this purpose, we use parametric representations that implement the expected scaling and analytic properties. We set [35]

$$\begin{aligned} M &= m_0 R^\beta m(\theta)\,, \\ t &= R(1-\theta^2)\,, \\ H &= h_0 R^{\beta\delta} h(\theta)\,, \end{aligned} \tag{3.1}$$

where h_0 and m_0 are normalization constants. The variable R is nonnegative and measures the distance from the critical point in the (t, H) plane, while the variable θ parametrizes

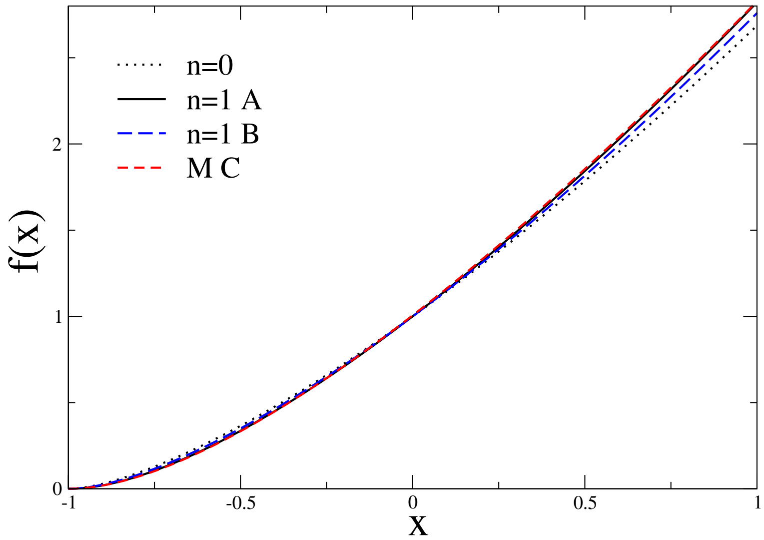

Figure 1: The scaling function $f(x)$ as obtained by the $n = 0$, $n = 1$ (A), and $n = 1$ (B) approximations. For comparison, we also show the Monte Carlo (MC) result of ref. [16], although it can be hardly distinguished from the $n = 1$ (A) curve.

the displacement along the lines of constant R. The functions $m(\theta)$ and $h(\theta)$ are odd and regular at $\theta = 0$ and at $\theta = 1$. The constants m_0 and h_0 can be chosen so that $m(\theta) = \theta + O(\theta^3)$ and $h(\theta) = \theta + O(\theta^3)$. The smallest positive zero θ_0 of $h(\theta)$, which should satisfy $\theta_0 > 1$, corresponds to the coexistence curve, i.e., to $T < T_c$ and $H \to 0$. The parametric representation satisfies the requirements of regularity of the equation of state. Singularities can only appear at the coexistence curve (due, for example, to the logarithms discussed in ref. [31]), i.e., for $\theta = \theta_0$. The mapping (3.1) is not invertible when its jacobian vanishes, which occurs when

$$Y(\theta) \equiv (1-\theta^2)m'(\theta) + 2\beta\theta m(\theta) = 0\,. \tag{3.2}$$

Thus, parametric representations based on the mapping (3.1) are acceptable only if $\theta_0 < \theta_l$, where θ_l is the smallest positive zero of the function $Y(\theta)$.

The functions $m(\theta)$ and $h(\theta)$ are related to the scaling function $f(x)$ through

$$\begin{aligned} x &= \frac{1-\theta^2}{\theta_0^2-1}\left[\frac{m(\theta_0)}{m(\theta)}\right]^{1/\beta}\,, \\ f(x) &= \left[\frac{m(\theta)}{m(1)}\right]^{-\delta}\frac{h(\theta)}{h(1)}\,. \end{aligned} \tag{3.3}$$

The asymptotic behavior (2.8) is reproduced by requiring that

$$h(\theta) \sim (\theta_0-\theta)^2 \qquad \text{for } \theta \to \theta_0\,. \tag{3.4}$$

Following ref. [23], we construct approximate polynomial parametric representations that have the expected singular behavior at the coexistence curve (Goldstone singularity) and match the known terms of the small-z expansion of $F(z)$, cf. eq. (2.22). We consider two distinct approximation schemes. In the first one, which we denote by (A), $h(\theta)$ is a fifth-order polynomial with a double zero at θ_0 and $m(\theta)$ is a polynomial of order $(1+2n)$:

$$\text{scheme (A)} : \qquad m(\theta) = \theta \left(1 + \sum_{i=1}^{n} c_i \theta^{2i}\right),$$
$$h(\theta) = \theta \left(1 - \frac{\theta^2}{\theta_0^2}\right)^2 . \tag{3.5}$$

In the second scheme, denoted by (B), we set

$$\text{scheme (B)} : \qquad m(\theta) = \theta \,,$$
$$h(\theta) = \theta \left(1 - \frac{\theta^2}{\theta_0^2}\right)^2 \left(1 + \sum_{i=1}^{n} c_i \theta^{2i}\right). \tag{3.6}$$

Here $h(\theta)$ is a polynomial of order $5+2n$ with a double zero at θ_0. For $n = 0$ approximations (A) and (B) coincide. It is worth noting that the $n = 0$ approximation becomes exact in $O(N)$ models for $N \to \infty$ [14, 23]. In both schemes the parameter θ_0 and the n coefficients c_i are determined by matching the small-z expansion of $F(z)$. The scaling function $F(z)$ can be written in terms of the parametric representation as

$$z = \rho\, m(\theta) \left(1 - \theta^2\right)^{-\beta} ,$$
$$F(z(\theta)) = \rho \left(1 - \theta^2\right)^{-\beta\delta} h(\theta) \,, \tag{3.7}$$

where ρ is a normalization [21, 26], which is fixed by matching the expansion (2.22) to order z^3. Explicitly, we have

$$\rho^2 = 6\beta(\delta - 1) - \frac{12}{\theta_0^2} - 6bc_1 \,, \tag{3.8}$$

where $b = 1, -1$ respectively for scheme (A) and (B). In both schemes, in order to fix θ_0 and the n coefficients c_i we need to know the values of $n + 1$ coefficients r_{2j}, i.e., $r_6, \ldots r_{6+2n}$. This method has been already applied to the determine the critical equation of state of the XY [23, 36] and Heisenberg [37] universality classes.

3.2 Results

In order to implement the above-presented approximation schemes, we use the Monte Carlo estimates [25] $\nu = 0.749(2)$ and $\eta = 0.0365(10)$ for the critical exponents, and the field-theoretical estimates $r_6 = 1.79(2)$ and $r_8 = 0.2(4)$ which take into account both fixed-dimension and ϵ-expansion results, see section 2.2. This will provide three different approximations: $n = 0$, $n = 1$ (A), and $n = 1$ (B). We find

$$\theta_0^2 = 2.795(40) \qquad n = 0 \,, \tag{3.9}$$

and

$$\theta_0^2 = 2.949(150) \,, \qquad c_1 = -0.0225(200) \,, \qquad n = 1 \text{ (A)} \,, \tag{3.10}$$
$$\theta_0^2 = 2.373(200) \,, \qquad c_1 = 0.0650(300) \,, \qquad n = 1 \text{ (B)} \,, \tag{3.11}$$

JHEP07(2003)029

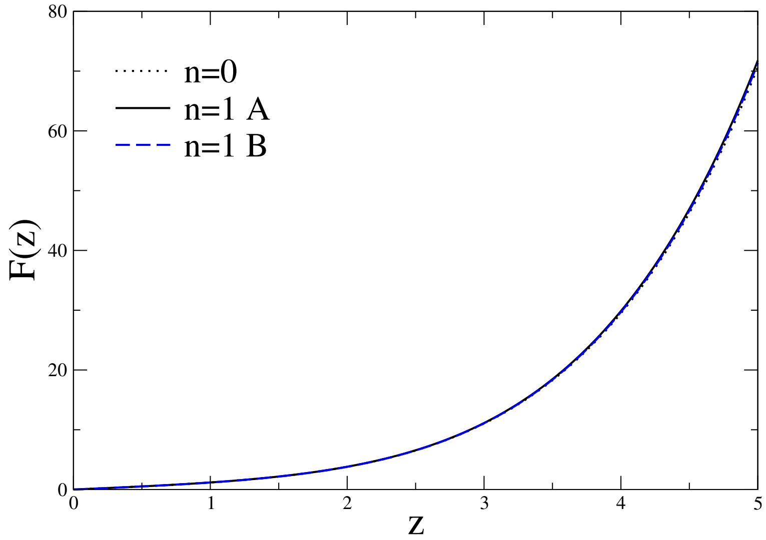

Figure 2: The scaling function $F(z)$.

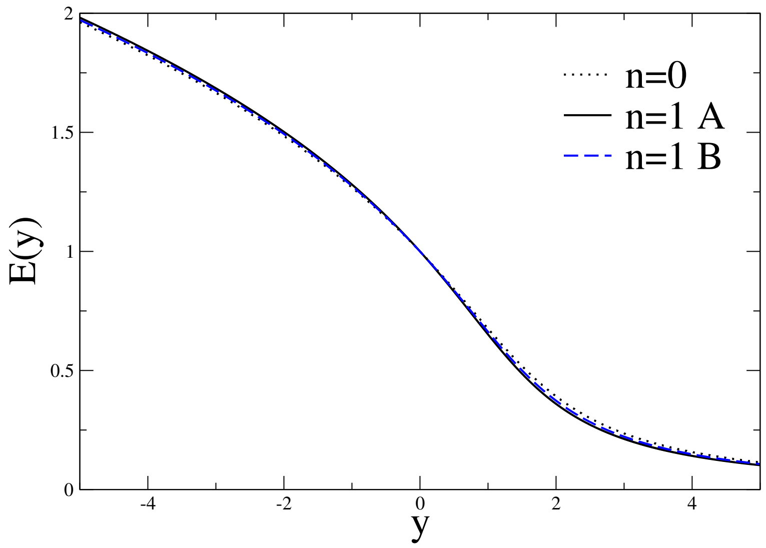

Figure 3: The scaling function $E(y)$.

where the number between parentheses indicates how much the coefficients vary when the input parameters change by one error bar. The relatively small value of c_1 in the $n = 1$ approximations supports the effectiveness of the approximation schemes. In figures 1, 2, 3, and 4 we show respectively the scaling functions $f(x)$, $F(z)$, $E(y)$, $D(y)$, as obtained from the $n = 0$, $n = 1$ (A), and $n = 1$ (B) approximations for the central values of the input parameters. The differences among the three approximations give an indication of the size of the systematic error of the approximation schemes. The figures show that it is rather small, suggesting that the $n = 0, 1$ representations already provide good approximations of the critical equation of state. In particular, the differences between the two $n = 1$ curves are within the uncertainty caused by the errors on the input parameters.

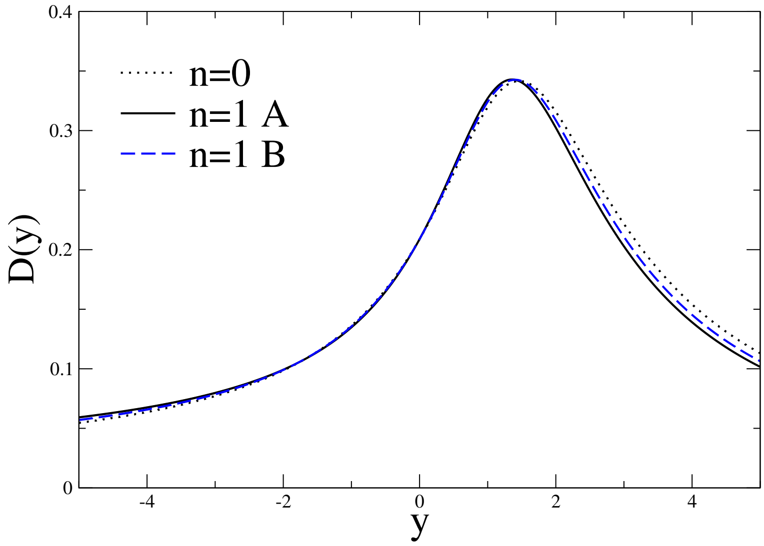

Figure 4: The scaling function $D(y)$.

In figure 1 we compare our results with the scaling function $f(x)$ obtained in ref. [16] by interpolating Monte Carlo data for the O(4) spin model (1.1). This comparison should be done with some caution, since Monte Carlo results are subject to scaling corrections and finite-size effects. Moreover, ref. [16] used $\delta = 4.86$ and $\beta = 0.38$, while we use the more precise estimates [25] $\delta = 4.789(6)$ and $\beta = 0.388(1)$. The Monte Carlo determination of $f(x)$ turns out to be in good agreement with our curves. Apparently, the best agreement is observed with the $n = 1$ (A) curve, but also the $n = 1$ (B) curve is substantially consistent. A more direct comparison with the Monte Carlo data of ref. [16] is presented in figure 5, where we plot $M/H^{1/\delta}$ versus $(J - J_c)/H^{1/\beta\delta}$ using $\delta = 4.789(6)$, $\beta = 0.388(1)$ and the raw data reported there. To minimize the finite-size and scaling corrections, we only consider the data for the largest lattices that are closest to the critical point, in particular those satisfying $H \leq 0.01$ and $|J - J_c| < 0.05$. The full line shown in figure 5 is the $n = 1$ (A) approximation, cf. eqs. (3.5) and (3.10), while the dotted lines show the uncertainty due to the error on the corresponding input parameters. The necessary normalization conditions related to the amplitudes B and B_c, cf. eq. (1.2), have been determined by fitting the data. The agreement is good: most Monte Carlo data are within the uncertainty of the $n = 1$ (A) curve. Substantial consistency is also found using the $n = 1$ (B) approximation.

In table 1 we report results concerning the behavior of the scaling functions $f(x)$ and $F(z)$ for $H = 0$ and on the critical isotherm, cf. eqs. (2.5), (2.6), (2.8), (2.19), and (2.24). We also report estimates of $y_{\max}$ and $D(y_{\max})$, where $y_{\max}$ is the value of y corresponding to the maximum of the scaling function $D(y)$ defined in eq. (2.9). The displayed errors refer only to the uncertainty of the input parameters and do not include the systematic error of the procedure, which may be determined by comparing the results of the various approximations. Our final estimates are reported in the table 1. Since we do not have theoretical reasons to prefer one of the approximation schemes, they are obtained by taking the average of the results $n = 1$ (A) and $n = 1$ (B). The error is just indicative: it is the sum

	$n=0$	$n=1$ (A)	$n=1$ (B)	final estimates
f_0^1	1.44(1)	1.53(9)	1.48(3)	1.50(8)
f_0^2	0.29(1)	0.35(6)	0.32(2)	0.33(5)
f_0^3	$-0.060(4)$	$-0.08(1)$	$-0.07(1)$	$-0.08(2)$
f_0^∞	0.82(1)	0.92(10)	0.87(3)	0.89(9)
f_2^{coex}	4.8(3)	2.2(1.3)	3.3(6)	2.8(1.4)
r_8	$-0.3(1)$	*0.2(4)	*0.2(4)	
r_{10}	2.7(5)	$-4(4)$	$-6(7)$	$-5(6)$
F_0^∞	0.0236(3)	0.0241(5)	0.0240(4)	0.0240(5)
$y_{\max}$	1.46(1)	1.35(10)	1.40(4)	1.38(9)
$D(y_{\max})$	0.3417(4)	0.3429(11)	0.3426(7)	0.3427(10)

Table 1: Results concerning the scaling functions obtained by using the $n = 0$, $n = 1$ (A), and $n = 1$ (B) approximations. Numbers marked with an asterisk are inputs, not predictions.

JHEP07(2003)029

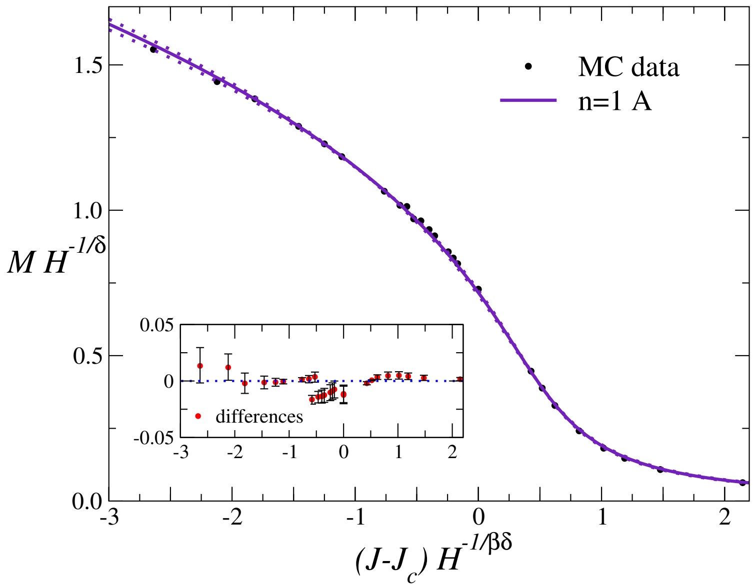

Figure 5: Monte Carlo data for $M/H^{1/\delta}$ versus $(J - J_c)/H^{1/\beta\delta}$ from ref. [16] compared with the $n = 1$ (A) approximation (full line). The dotted lines show the uncertainty on the curve due to the errors on the input parameters. The inset shows the differences between the Monte Carlo data and the $n = 1$ (A) curve: the errors are related to the uncertainty on the theoretical curve; the statistical errors of the Monte Carlo data are much smaller.

of the uncertainty induced by the input parameters (we take the average of the two errors) and of half of the difference between the two approximations. In most cases this procedure leads to an estimate that includes the $n = 0$ result within its error. For comparison, we also report the estimates $f_2^{\text{coex}} = 2.20(5)$, $f_0^\infty = 0.888(7)$, $F_0^\infty = 0.018(4)$, $y_{\max} = 1.33(5)$, and $D(y_{\max}) = 0.341(1)$, which can be obtained from the results of the fits of the Monte Carlo data reported in refs. [16, 17]. Finally, we mention that the result for r_{10} is consistent with the estimate $r_{10} = 9(17)$ obtained from the analysis of its $O(\epsilon^3)$ series [18].

4. Universal amplitude ratios

Universal amplitude ratios characterize the critical behavior of thermodynamic quantities that do not depend on the normalizations of the external (magnetic) field, of the order parameter (magnetization), and of the temperature. From the scaling function $f(x)$ one may derive many universal amplitude ratios involving zero-momentum quantities, such as the specific heat, the magnetic susceptibility, etc.... For example, the universal ratio of the specific-heat amplitudes in the two phases can be written as

$$\frac{A^+}{A^-} = \frac{\varphi(\infty)}{\varphi(-1)} \tag{4.1}$$

where, in the case of the O(4) universality class for which $-1 < \alpha < 0$,

$$\varphi(x) = \frac{x|x|^{\alpha-2} f'(0)}{\alpha - 1} + \frac{|x|^\alpha f''(0)}{\alpha} - |x|^{\alpha-2} f(x) + \int_0^x dy |y|^{\alpha-2} \left[f'(y) - f'(0) - y f''(0) \right] . \tag{4.2}$$

Several universal amplitude ratios can be expressed in terms of the amplitudes derived from the singular behavior of the specific heat

$$C_H = A^\pm |t|^{-\alpha} + b\,, \tag{4.3}$$

where b is a nonuniversal constant, the magnetic susceptibility in the high-temperature phase

$$\chi = N C^+ t^{-\gamma} \tag{4.4}$$

with $N = 4$, the zero-momentum four-point connected correlation function in the high-temperature phase

$$\chi_4 = \frac{N(N+2)}{3} C_4^+ t^{-\gamma-2\beta\delta} \tag{4.5}$$

(again with $N = 4$), the second-moment correlation length in the high-temperature phase, corresponding to the inverse mass in the zero-momentum renormalization scheme, cf. eqs. (2.12) and (2.13),

$$\xi = f^+ t^{-\nu}\,, \tag{4.6}$$

the spontaneous magnetization on the coexistence curve, cf. eqs. (2.2) and (2.3). We also consider amplitudes along the crossover line $t_{\rm max}(H)$, that is defined as the reduced temperature where the longitudinal magnetic susceptibility $\chi_L(t, H) = \partial M/\partial H$ has a maximum at H fixed, i.e.

$$t_{\rm max}(H) = T_p H^{1/(\gamma+\beta)}\,, \tag{4.7}$$

$$\chi_L(t_{\rm max}, H) = C_p t_{\rm max}^{-\gamma}\,. \tag{4.8}$$

In table 2 we report the definitions of several universal amplitude ratios that have been considered in the literature. Their expressions in terms of the functions $m(\theta)$ and $h(\theta)$ can be found in ref. [37]. Note the following relations

$$\begin{aligned} f_0^\infty &= R_\chi^{-1}\,, & f_1^\infty &= \frac{R_4^+}{6R_\chi}\,, \\ y_{\rm max} &= P_m^{1/\beta}\,, & D(y_{\rm max}) &= R_p^{-1} P_m^{1-\delta} R_\chi\,. \end{aligned} \tag{4.9}$$

where f_0^∞ and f_1^∞ are related to the large-x behavior of $f(x)$, cf. eq. (2.6), and $y_{\rm max}$ is the value of y where the scaling function $D(y)$ takes its maximum.

JHEP07(2003)029

Universal Amplitude Ratios	
$U_0 \equiv A^+/A^-$	$R_c^+ \equiv \alpha A^+ C^+/B^2$
$R_4^+ \equiv -C_4^+ B^2/(C^+)^3$	$R_\chi \equiv C^+ B^{\delta-1}/B_c^\delta$
$g_4 \equiv -C_4^+/[(C^+)^2(f^+)^3]$	$Q^+ \equiv \alpha A^+(f^+)^3$
$R_\xi^+ \equiv (Q^+)^{1/3}$	$Q_c \equiv B^2(f^+)^3/C^+$
$P_m \equiv T_p^\beta B/B^c$	$R_p \equiv C^+/C_p$

Table 2: Definitions of several universal amplitude ratios.

	$n=0$	$n=1$ (A)	$n=1$ (B)	final estimates
U_0	2.03(3)	1.89(11)	1.93(6)	1.91(10)
R_c^+	0.250(5)	0.274(25)	0.263(9)	0.27(2)
R_4^+	8.04(7)	7.5(5)	7.7(2)	7.6(4)
R_χ	1.22(2)	1.09(12)	1.15(4)	1.12(11)
P_m	1.159(3)	1.12(3)	1.140(11)	1.13(2)
R_p	2.049(3)	2.041(7)	2.042(6)	2.042(7)
$R_c^+ R_4^+$	2.01(4)	2.05(5)	2.03(3)	2.04(5)

Table 3: Estimates of several universal amplitude ratios.

In table 3 we report the estimates of several universal amplitude ratios, as derived by using the approximations $n = 0$, $n = 1$ (A), and $n = 1$ (B). Again, the displayed errors refer only to the uncertainty of the input parameters. Our final estimates are determined as in the case of the quantities reported in table 1. They can be again compared with the Monte Carlo results reported in refs. [16, 17]: $R_\chi = 1.126(9)$, $R_4^+ = 8.6(9)$ and $P_m = 1.11(2)$. There is good agreement, keeping again into account that refs. [16, 17] used different values of the critical exponents. Moreover, using eq. (4.1) and the approximate interpolation formula of refs. [16, 17] we obtain $U_0 = 2.0(2)$, where the error is obtained by varying the curve parameters within their error. We also mention the estimates [38] $U_0 \approx 2.044$ and $R_c^+ \approx 0.263$, obtained by an analysis of three-loop series in the minimal-subtraction scheme without ϵ expansion. They are in good agreement with those presented in table 3.

Universal amplitude ratios involving the correlation-length amplitude f^+, such as R_ξ^+ and Q_c, can be obtained by using the estimate of g_4. If we take [24] $g_4 = 17.30(6)$, we obtain

$$R_\xi^+ = \left(\frac{R_c^+ R_4^+}{g_4}\right)^{1/3} = 0.490(4)\,, \tag{4.10}$$

$$Q_c = \frac{R_4^+}{g_4} = 0.44(2)\,. \tag{4.11}$$

Acknowledgments

We thank Luigi Del Debbio for useful and interesting discussions.

References

[1] R.D. Pisarski and F. Wilczek, *Remarks on the chiral phase transition in chromodynamics*, *Phys. Rev.* **D 29** (1984) 338.

[2] F. Wilczek, *Application of the renormalization group to a second order QCD phase transition*, *Int. J. Mod. Phys.* **A 7** (1992) 3911.

[3] K. Rajagopal and F. Wilczek, *Static and dynamic critical phenomena at a second order QCD phase transition*, *Nucl. Phys.* **B 399** (1993) 395 [hep-ph/9210253].

[4] K. Rajagopal, *The chiral phase transition in QCD: critical phenomena and long wavelength pion oscillations*, hep-ph/9504310.

[5] F. Wilczek, *QCD in extreme conditions*, hep-ph/0003183.

[6] A. Butti, A. Pelissetto and E. Vicari, *On the nature of the finite-temperature transition in QCD*, hep-ph/0307036.

[7] CP-PACS collaboration, A. Ali Khan et al., *Phase structure and critical temperature of two flavor QCD with renormalization group improved gauge action and clover improved Wilson quark action*, *Phys. Rev.* **D 63** (2001) 034502 [hep-lat/0008011]; *Equation of state in finite-temperature QCD with two flavors of improved Wilson quarks*, *Phys. Rev.* **D 64** (2001) 074510 [hep-lat/0103028].

[8] Y. Iwasaki, K. Kanaya, S. Kaya and T. Yoshie, *Scaling of chiral order parameter in two-flavor QCD*, *Phys. Rev. Lett.* **78** (1997) 179 [hep-lat/9609022].

[9] F. Karsch, E. Laermann and A. Peikert, *Quark mass and flavor dependence of the QCD phase transition*, *Nucl. Phys.* **B 605** (2001) 579 [hep-lat/0012023].

[10] J.B. Kogut and D.K. Sinclair, *Scaling behavior at the $N_t = 6$ chiral phase transition for 2-flavor lattice QCD with massless staggered quarks, and an irrelevant 4-fermion interaction*, *Phys. Rev.* **D 64** (2001) 034508 [hep-lat/0104011]; *The universality class of the finite temperature transition in two flavor quantum chromodynamics*, hep-lat/0005007.

[11] E. Laermann and O. Philipsen, *Status of lattice QCD at finite temperature*, hep-ph/0303042.

[12] F. Karsch and E. Laermann, *Thermodynamics and in-medium hadron properties from lattice QCD*, hep-lat/0305025.

[13] E. Brézin, D.J. Wallace and K.G. Wilson, *Feynman-graph expansion for the equation of state near the critical point (Ising-like case)*, *Phys. Rev. Lett.* **29** (1972) 591; *Feynman-graph expansion for the equation of state near the critical point*, *Phys. Rev.* **B 7** (1973) 232.

[14] E. Brézin and D.J. Wallace, *Critical behavior of a classical Heisenberg ferromagnet with many degrees of freedom*, *Phys. Rev.* **B 7** (1973) 1967

[15] D. Toussaint, *Scaling functions for $O(4)$ in three dimensions*, *Phys. Rev.* **D 55** (1997) 362 [hep-lat/9607084].

[16] J. Engels and T. Mendes, *Goldstone-mode effects and scaling function for the three-dimensional $O(4)$ model*, *Nucl. Phys.* **B 572** (2000) 289 [hep-lat/9911028].

[17] J. Engels, S. Holtmann, T. Mendes and T. Schulze, *Finite-size-scaling functions for 3d $O(4)$ and $O(2)$ spin models and QCD*, *Phys. Lett.* **B 514** (2001) 299 [hep-lat/0105028].

JHEP07(2003)029

[18] A. Pelissetto and E. Vicari, *The effective potential of N-vector models: a field-theoretic study to $O(\epsilon^3)$*, *Nucl. Phys.* **B 575** (2000) 579 [cond-mat/9911452]; *The effective potential in three-dimensional $O(N)$ models*, *Nucl. Phys.* **B 522** (1998) 605 [cond-mat/9801098]; *Four-point renormalized coupling constant and Callan-Symanzik β-function in $O(N)$ models*, *Nucl. Phys.* **B 519** (1998) 626 [cond-mat/9711078].

[19] A.I. Sokolov, E.V. Orlov, V.A. Ulkov and S.S. Kashtanov, *Renormalized coupling constants for the three-dimensional N-vector model from field theory*, hep-th/9810082.

[20] J. Zinn-Justin, *Quantum field theory and critical phenomena*, 4^{th} edition, Clarendon Press, Oxford 2001.

[21] R. Guida and J. Zinn-Justin, *3d Ising model: the scaling equation of state*, *Nucl. Phys.* **B 489** (1997) 626 [hep-th/9610223].

[22] M. Campostrini, A. Pelissetto, P. Rossi and E. Vicari, *Improved high-temperature expansion and critical equation of state of three-dimensional Ising-like systems*, *Phys. Rev.* **E 60** (1999) 3526 [cond-mat/9905078]; *25th-order high-temperature expansion results for three-dimensional Ising-like systems on the simple cubic lattice*, *Phys. Rev.* **E 65** (2002) 066127 [cond-mat/0201180].

[23] M. Campostrini, A. Pelissetto, P. Rossi and E. Vicari, *The critical equation of state of three-dimensional XY systems*, *Phys. Rev.* **B 62** (2000) 5843 [cond-mat/0001440].

[24] R. Guida and J. Zinn-Justin, *Critical exponents of the N-vector model*, *J. Phys.* **A 31** (1998) 8103 [cond-mat/9803240].

[25] M. Hasenbusch, *Eliminating leading corrections to scaling in the three-dimensional $O(N)$ symmetric Φ^4 model: $N = 3$ and $N = 4$*, *J. Phys.* **A 34** (2001) 8221 [cond-mat/0010463].

[26] A. Pelissetto and E. Vicari, *Critical phenomena and renormalization-group theory*, *Phys. Rept.* **368** (2002) 549 [cond-mat/0012164].

[27] E. Brézin and J. Zinn-Justin, *Spontaneous breakdown of continuous symmetries near two dimensions*, *Phys. Rev.* **B 14** (1976) 3110.

[28] I.D. Lawrie, *Goldstone modes and coexistence in isotropic N vector models*, *J. Phys.* **A 14** (1981) 2489.

[29] D.J. Wallace and R.K.P. Zia, *Singularities induced by Goldstone modes*, *Phys. Rev.* **B 12** (1975) 5340.

[30] L. Schäfer and H. Horner, *Goldstone mode singularities and equation of state of an isotropic magnet*, *Z. Phys.* **B29** (1978) 251.

[31] A. Pelissetto and E. Vicari, *Low-temperature effective potential of the Ising model*, *Nucl. Phys.* **B 540** (1999) 639 [cond-mat/9805317].

[32] G. Parisi, *Field-theoretic approach to second-order phase transitions in two- and three-dimensional systems*, Cargèse Lectures (1973), *J. Stat. Phys.* **23** (1980) 49.

[33] C. Itzykson and J.M. Drouffe, *Statistical field theory*, Cambridge University Press, Cambridge 1989.

[34] G.A. Baker, B.G. Nickel, M.S. Green and D.I. Meiron, *Ising model critical indices in three-dimensions from the Callan-Symanzik equation*, *Phys. Rev. Lett.* **36** (1976) 1351; G.A. Baker, Jr., B.G. Nickel, and D.I. Meiron, *Critical indices from perturbation analysis of the Callan-Symanzik equation*, *Phys. Rev.* **B 17** (1978) 1365.

JHEP07(2003)029

[35] P. Schofield, *Parametric representation of the equation of state near a critical point*, *Phys. Rev. Lett.* **22** (1969) 606;
B.D. Josephson, *Equation of state near the critical point*, *J. Phys.* **C 2** (1969) 1113;
P. Schofield, J.D. Litster, and J.T. Ho, *Correlation between critical coefficients and critical exponents*, *Phys. Rev. Lett.* **23** (1969) 1098.

[36] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi and E. Vicari, *Critical behavior of the three-dimensional XY universality class*, *Phys. Rev.* **B 63** (2001) 214503 [cond-mat/0010360].

[37] M. Campostrini, M. Hasenbusch, A. Pelissetto, P. Rossi and E. Vicari, *Critical exponents and equation of state of the three-dimensional Heisenberg universality class*, *Phys. Rev.* **B 65** (2002) 144520 [cond-mat/0110336].

[38] H. Kleinert and B. Van den Bossche, *Three-loop critical exponents, amplitude functions, and amplitude ratios from variational perturbation theory*, *Phys. Rev.* **E 63** (2001) 056113 [cond-mat/0011329].